UNIT-2

(Lecture-3)

Design of Infinite Impulse Response Digital Filters: Impulse Invariant Transformation

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Impulse Invariant Transformation

In this technique, the desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter. That is ,

$h(n) = h_a(nT)$ -----------(1)

Where T is the sampling interval.

Consider a simple distinct pole case for the analog filter's system

$$H_a(s) = \sum_{i=1}^{M} \frac{A_i}{s - p_i}$$ -------------(2)

Impulse Invariant Transformation

The impulse response of the system is specified by the equation(2) can be obtained by taking the inverse Laplace transform and it will be of the form

$$h_a(t) = \sum_{i=1}^{M} A_i e^{p_i t} u_a(t)$$ -------------(3)

Where $u_a(t)$ is the unit step function in continuous time. The impulse response h(n) of the equivalent digital filter is obtained by uniformly sampling $h_a(t)$, i.e. by applying Eq. (1)

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i e^{p_i n t} u_a(nT)$$ --------------(4)

Impulse Invariant Transformation

The system response of the digital system of equation(4) can be obtain by taking the z-transform, i.e.

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

Using Eq.(4)

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{M} A_i\, e^{p_i n T}\, u_a(nT) \right] z^{-n} \quad \text{------------(5)}$$

Interchanging the order of summation,

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i\, e^{p_i n T}\, u_a(nT) \right] z^{-n}$$

$$H(z) = \sum_{i=1}^{M} \frac{A_i}{1 - e^{p_i T}\, z^{-1}} \quad \text{------------------(6)}$$

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Impulse Invariant Transformation

Now, by comparing equation(2) and equation(6) the mapping formula for the impulse invariant transformation is given by

$$\frac{1}{s - p_i} \longrightarrow \frac{1}{1 - e^{p_i T} z^{-1}} \quad \text{-------------(7)}$$

Equation 7 shows that the analog pole at $s = p_i$ is mapped into a digital pole at $z = e^{p_i T}$. Therefore the analog poles and the digital poles are related by the relations

$$z = e^{sT} \quad \text{-------------(8)}$$

The general characteristics of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$

Therefore, $re^{j\omega} = e^{\sigma T} e^{j\Omega T}$

Clearly, $r = e^{\sigma T}$ and $\omega = \Omega T$

Impulse Invariant Transformation

Some of the properties of the impulse invariant transformation are given below:-

$$\frac{1}{(s+s_i)^m} \rightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[\frac{1}{1-e^{-sT} z^{-1}}\right]; s \rightarrow s_i \qquad \text{---------(9)}$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1}+e^{-2aT} z^{-2}} \qquad \text{--------(10)}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1}+e^{-2aT} z^{-2}} \qquad \text{--------(11)}$$

Impulse Invariant Transformation

Example: **Convert the analog filter into a digital filter whose system function is** $H(s) = \frac{s + 0.2}{(s + 0.2)^2 + 9}$

Use the impulse invariant technique. Assume T = 1s.

Solution:

The system response of the analog filter is of the standard form

$$H(s) = \frac{s + a}{(s + a)^2 + b^2}$$

Where a = 0.2 and b = 3. The system response of the digital filter can be obtained using equation (10)

$$H(z) = \frac{1 - e^{-aT}(\cos bT)\, z^{-1}}{1 - 2e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT} z^{-2}}$$

$$= \frac{1 - e^{-0.2T}(\cos 3T)\, z^{-1}}{1 - 2e^{-0.2T}(\cos 3T)\, z^{-1} + e^{-0.4T} z^{-2}}$$

Impulse Invariant Transformation

Taking T = 1s,

$$H(z) = \frac{1-(0.8187)(-0.99)z^{-1}}{1-2(0.8187)(-0.99)z^{-1}+0.6703z^{-2}}$$

$$H(z) = \frac{1+(0.8105)z^{-1}}{1+1.6210z^{-1}+0.6703z^{-2}}$$